Theory of Impartial Games

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Based on http://web.mit.edu/sp.268/www/nim.pdf

February 9, 2017

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Overview



Definition

• Examples

Nim

- Rules
- Example of play
- Types of positions
- Properties of positions

3 Nimber

- XOR
- Nimber for Nim
- Theorem
- Example

4 Summary

Definition

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• two players

- two players
- moving alternately

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- two players
- moving alternately
- finite set of states

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Image: Image:

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- two players
- moving alternately
- finite set of states
- finite play

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- two players
- moving alternately
- finite set of states
- finite play
- player who can't make a move loses

- two players
- moving alternately
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- player who can't make a move loses
- possible moves are the same for each player

- two players
- moving alternately
- finite set of states
- finite play
- player who can't make a move loses
- possible moves are the same for each player

IMPARTIAL GAME

Examples

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• Nim

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• Nim

• Subtraction game

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- Nim
- Subtraction game

Impartial games are not

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- Nim
- Subtraction game

Impartial games are not

checkers

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- Nim
- Subtraction game

Impartial games are not

- checkers
- chess

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- Nim
- Subtraction game

Impartial games are not

- checkers
- chess
- GO

Image: A math and A math and

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• piles of stones

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- piles of stones
- select one pile

- piles of stones
- select one pile
- decrease amount of stones in chosen pile

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Image: A math and A

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Image: A matrix and a matrix

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Types of positions

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State in which current player can win regardless of opponent's moves.

State in which current player can win regardless of opponent's moves.

Losing position

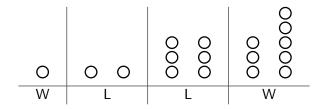
State in which current player can't win if the opponent plays optimally.

State in which current player can win regardless of opponent's moves.

Losing position

State in which current player can't win if the opponent plays optimally.

Examples:



Properties of positions

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leads to at least one losing position

leads to at least one losing position

Losing position

leads to winning positions only

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Ope	Operation array											
\oplus	0	1										
0	0	1										
1	1	0										

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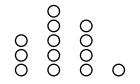
Ope	ratio	on a	rray					
\oplus	0	1						
0	0	1						
1	1	0						

Example:

$$\begin{array}{cccc}
6 \oplus 3 = 5 \\
 & 1 & 1 & 0 \\
 \hline
\oplus & 0 & 1 & 1 \\
\hline
 & 1 & 0 & 1
\end{array}$$

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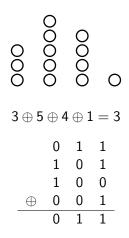
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Theorem

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Theorem

Position is winning if and only if its nimber is non-zero.

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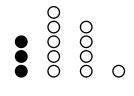
 $3 \oplus 5 \oplus 4 \oplus 1 = 3$

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 $\mathbf{3}\oplus\mathbf{5}\oplus\mathbf{4}\oplus\mathbf{1}=\mathbf{3}$

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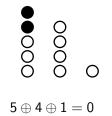
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O O O O O O $3 \oplus 4 \oplus 1 = 6$

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O O O O $2 \oplus 2 \oplus 1 = 1$

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O O O O $2 \oplus 2 \oplus 1 = 1$

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$\begin{array}{c} O \\ O \\ 2 \oplus 2 = 0 \end{array}$

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2 = 2

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First player has won!

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Image: A matrix

Summary

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Thanks for attention

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