Theory of Impartial Games

Jakub Cisło

Jagiellonian University

jakub@cislo.net.pl

Based on
http://web.mit.edu/sp.268/www/nim.pdf

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Overview

Definition

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- two players

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- two players
- moving alternately

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IMPARTIAL GAME

Examples

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Impartial games <u>are not</u>

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- chess

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Rules of Nim

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- piles of stones

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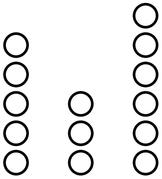
- select one pile

Rules of Nim

- piles of stones
- select one pile
- decrease amount of stones in chosen pile

Example of play



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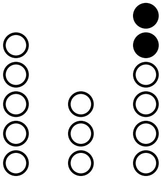

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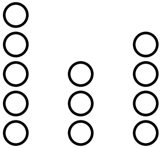

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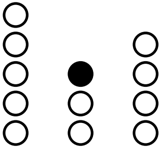

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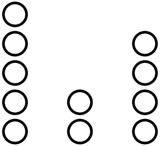

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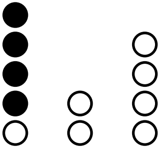

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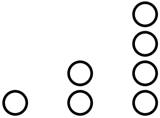

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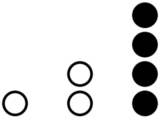

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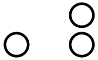

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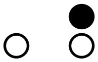

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Types of positions

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Winning position

State in which current player can win regardless of opponent's moves.

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State in which current player can't win if the opponent plays optimally.

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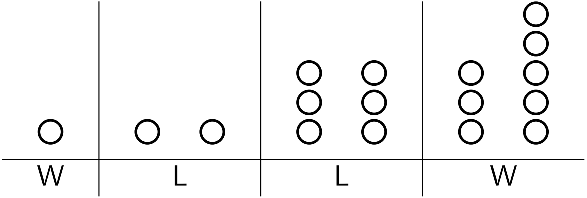

Properties of positions

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Winning position

leads to at least one losing position

Properties of positions

Winning position

leads to at least one losing position

Losing position

leads to winning positions only

XOR

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Operation array

$\oplus$	0	1
0	0	1
1	1	0

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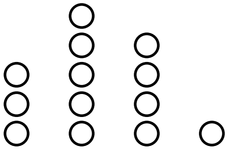
0	0	1
1	1	0

Example:

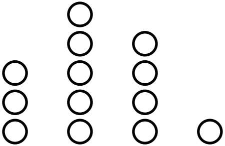
$$6 \oplus 3 = 5$$

$$\begin{array}{cccc} & 1 & 1 & 0 \\ \oplus & 0 & 1 & 1 \\ \hline & 1 & 0 & 1 \end{array}$$

Nimber for Nim



Nimber for Nim



$$3 \oplus 5 \oplus 4 \oplus 1 = 3$$

$$\begin{array}{cccc} & 0 & 1 & 1 \\ & 1 & 0 & 1 \\ & 1 & 0 & 0 \\ \oplus & 0 & 0 & 1 \\ \hline & 0 & 1 & 1 \end{array}$$

Theorem

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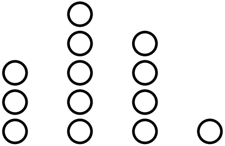
Position is winning if and only if its nimber is non-zero.

Example



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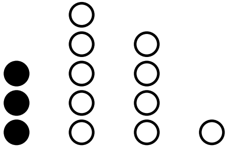

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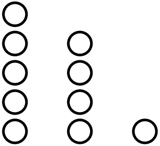

$$5 \oplus 4 \oplus 1 = 0$$

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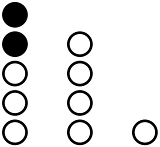

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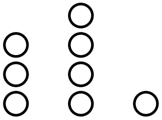
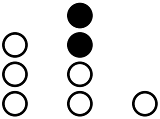


$$3 \oplus 4 \oplus 1 = 6$$

Example



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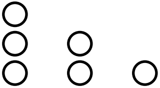
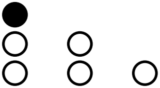


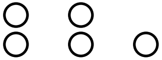
$$3 \oplus 2 \oplus 1 = 0$$

Example



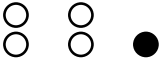
$$3 \oplus 2 \oplus 1 = 0$$

Example



$$2 \oplus 2 \oplus 1 = 1$$

Example



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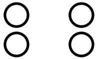
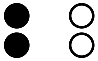


$$2 \oplus 2 = 0$$

Example



$$2 \oplus 2 = 0$$

Example

$$2 = 2$$

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Example

First player has won!

Summary

Thanks for attention